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ELECTRIC S-BRANE SOLUTIONS WITH A PARALLEL CHARGE DENSITY FORM ON A RICCI-FLAT FACTOR SPACE

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We generalize the previously studied cosmological solutions in D -dimensional gravity with an antisymmetric $(p+2)$ -form to the case when the spatial part of the metric is Ricci-flat rather than flat. These generalized solutions are characterized by a parallel self-dual or anti-self-dual charge density form Q of rank $2m$ and satisfy the condition $Q^2 > 0$. As with the previous flat-space case, these electric, S-brane solutions only exist when $D = 4m + 1 = 5, 9, 13, \dots$ and $p = 2m - 1 = 1, 3, 5, \dots$. We further generalize these solutions by adding Ricci-flat factor-spaces which are not covered by branes.

1. Introduction

In the paper, we continue the study of S -brane solutions with a maximum number of electric branes [1, 2, 3]. S -branes are brane solutions having “warp” factors which are time-dependent rather than spatially dependent. They are thought to play a role in higher-dimensional cosmological models, especially string theory inspired cosmologies. A sampling of the work done on S -brane solutions and their physical applications can be found in [4–18] and references therein.

In [1], electric S -brane solutions, with a maximum number of orthogonal branes, were constructed. These solutions only occurred for $D = 4m + 1$ spacetime dimensions with $(2m + 1)$ -forms. They also had the property that the charge density forms of the electric branes were either self-dual or anti-self-dual. For example, in the case when $D = 5$ with 3-forms, the six electric branes obeyed the following relations:

$$Q_{12} = \mp Q_{34}, \quad Q_{13} = \pm Q_{24}, \quad Q_{14} = \mp Q_{23}. \quad (1.1)$$

or, equivalently,

$$Q_{ij} = \pm \frac{1}{2} \varepsilon_{ijkl} Q^{kl} = \pm (*Q)_{ij}. \quad (1.2)$$

The solution (metric plus form fields and scalar fields) for the case when all the Q_{ij} , $i < j$, were non-zero, followed from the restrictions coming from the non-diagonal part of Hilbert-Einstein equations.

Some of the cosmological consequences of these maximal electric S -brane solutions (in case of the absence of scalar fields) were investigated in [2], e.g., suppression of oscillating behaviour near the singularity due to constraints coming from the diagonal form of the metric

was found. (This is important for the billiard approach for models with branes suggested in [19]).

In addition to their relation to cosmology, the study of electric S -branes with various charge densities may be of possible interest to string theory because of the important role played by the relations between the charge densities of D -branes [20]. Such relations in the general case may be described mathematically by K -theory (see [21, 22] and references therein). Here, as in our previous work [1, 2], the relationships between the charge densities follow directly from the equations of motion.

2. S-brane solution with flat factor space

2.1. D dimensional gravity and $(p+2)$ -form

As in [1], we consider the model governed by the action

$$S = \int_M d^D z \sqrt{|g|} \left[R[g] - \frac{1}{q!} F^2 \right], \quad (2.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric, $R[g]$ is the D -dimensional Ricci scalar calculated from the metric g , and

$$F = dA = \frac{1}{q!} F_{M_1 \dots M_q} dz^{M_1} \wedge \dots \wedge dz^{M_q}$$

is a q -form, $q = p + 2 \geq 1$. For $p = 0$ one would have a 2-form analogous to the Maxwell field strength tensor. Thus the action considered in (2.1) gives D -dimensional gravity coupled to an “electromagnetic” field. In (2.1), we denote $|g| = |\det(g_{MN})|$ and $F^2 = F_{M_1 \dots M_q} F_{N_1 \dots N_q} g^{M_1 N_1} \dots g^{M_q N_q}$. The equations of motion corresponding to (2.1) are

$$R_N^M - \frac{1}{2} \delta_N^M R = T_N^M, \quad (2.2)$$

$$\nabla_{M_1} [g] F^{M_1 \dots M_q} = 0. \quad (2.3)$$

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In (2.3), $\nabla[g]$ is the covariant derivative operator corresponding to g . Eqs. (2.2) and (2.3) are the multidimensional Einstein-Hilbert equations and the “Maxwell” equations for the q -form, respectively. The source term in (2.2), the stress-energy tensor of q -form, reads

$$T_N^M[F, g] = \frac{1}{q!} \left[-\frac{1}{2} \delta_N^M F^2 + q F_{N M_2 \dots M_q} F^{M M_2 \dots M_q} \right]. \quad (2.4)$$

2.2. Cosmological type solutions for $D = 5, 9, 13, \dots$

Here we give a quick summary of Sp -brane solutions from [2] in dimensions

$$D = n + 1 = 4m + 1 = 5, 9, 13, \dots \quad (2.5)$$

with $(p + 2)$ -forms whose rank is given by

$$p = 2m - 1 = 1, 3, 5, \dots \quad (2.6)$$

These solutions are defined on the manifold

$$M = (t_-, t_+) \times \mathbb{R}^n \quad (2.7)$$

and have the following form:

$$ds^2 = -e^{2n\phi(t)} dt^2 + e^{2\phi(t)} \sum_{i=1}^n (dy^i)^2, \quad (2.8)$$

$$F = e^{2f(t)} dt \wedge Q, \quad (2.9)$$

$$Q = \frac{1}{(p+1)!} Q_{i_0 \dots i_p} dy^{i_0} \wedge \dots \wedge dy^{i_p}. \quad (2.10)$$

$Q_{i_0 \dots i_p}$ are constant components of the charge density form Q . Q is self-dual or anti-self-dual in flat Euclidean space \mathbb{R}^n , i.e.,

$$Q_{i_0 \dots i_p} = \pm \frac{1}{(p+1)!} \varepsilon_{i_0 \dots i_p j_0 \dots j_p} Q^{j_0 \dots j_p} = \pm (*Q)_{i_0 \dots i_p}. \quad (2.11)$$

We now put

$$Q^2 \equiv \frac{1}{(p+1)!} \sum_{i_0, \dots, i_p} Q_{i_0 \dots i_p}^2 > 0, \quad (2.12)$$

i.e., at least one charge density is non-zero: $Q_{i_0 \dots i_p} \neq 0$, for some $i_0 < \dots < i_p$. Then it can be shown [1] that Eqs. (2.2) (2.3) reduce to the set of equations

$$\ddot{f} = \dot{f}^2 = -Q^2 K e^{2f}. \quad (2.13)$$

The dots denote derivatives with respect to t and

$$K \equiv -\frac{n}{4(n-1)}. \quad (2.14)$$

The function $\phi(t)$ is given in terms of $f(t)$ by

$$\phi(t) = \frac{2}{n} f(t). \quad (2.15)$$

Thus, the solution of the system is determined once (2.13) is solved. A solution for $f(t)$ is given by

$$f = -\ln \left[|t - t_0| K Q^2 |^{1/2} \right]. \quad (2.16)$$

The above solution describes a collection of $l \leq \frac{(4m)!}{(2m)!^2}$ electric Sp -branes.

3. Generalization to Ricci-flat factor space

Here we generalize the solution from the previous section to the case when the manifold (2.7) is replaced by the manifold

$$M = (t_-, t_+) \times N, \quad (3.1)$$

where N is an n -dimensional oriented manifold of dimension $n = 4m$, $m = 1, 2, \dots$, equipped with the Ricci-flat metric $h = h_{ij}(y) dy^i \otimes dy^j$ of Euclidean signature, i.e., the Ricci tensor calculated from h is zero, $R_{ij}[h] = 0$. Let

$$Q = \frac{1}{(p+1)!} Q_{i_0 \dots i_p}(y) dy^{i_0} \wedge \dots \wedge dy^{i_p} \quad (3.2)$$

be a form of rank $2m$ defined on the manifold N . The components $Q_{i_0 \dots i_p}(y)$ now depend on the spatial coordinates y^i . The form Q satisfies two requirements. First, it is parallel, i.e., covariantly constant w.r.t. h :

$$(i) \quad \nabla[h]Q = 0, \quad (3.3)$$

and, second, it is self-dual or anti-self-dual:

$$(ii) \quad Q = \pm * Q. \quad (3.4)$$

Here $* = *[h]$ is the Hodge operator corresponding to the metric h . From condition (i) one finds that

$$Q^2 \equiv \frac{1}{(p+1)!} h^{i_0 j_0} \dots h^{i_p j_p} Q_{i_0 \dots i_p} Q_{j_0 \dots j_p} \quad (3.5)$$

is constant. It also follows from (3.3) that Q is closed:

$$dQ = 0. \quad (3.6)$$

In what follows we put

$$Q^2 > 0, \quad (3.7)$$

or, equivalently, Q is non-zero.

For the metric and p -form, we take the following ansatz on the manifold (3.1) similar to the ansatz used in the flat case, (2.8), (2.9):

$$ds^2 = -e^{2n\phi(t)} dt^2 + e^{2\phi(t)} h_{ij}(y) dy^i dy^j, \quad (3.8)$$

$$F = e^{2f(t)} dt \wedge Q. \quad (3.9)$$

The $2m$ -form Q is defined in (3.2) and satisfies (3.3), (3.4) and (3.7). Except for the spatial metric h , this is similar to the previous flat case ansatz. In fact, we will find that the form of these new (“Ricci-flat”) solutions is essentially the same as in the flat case. K and the function $\phi(t)$ are again given by (2.14) and (2.15). As before, once $f(t)$ is given, the entire solution is obtained. The function $f(t)$ again obeys Eq. (2.13), and thus the solution is (2.16).

Now we show that the metric with the line element (3.8) and the $(2m + 1)$ -form (3.9) do satisfy the field equations (2.2), (2.3) for the case when the manifold is

given by (3.1). Because of (3.3), the “Maxwell” equations (2.3) reduce to those for the flat-case solution from the previous section, and so the $(p+2)$ -form part of the solution has the same form as in the flat case.

The gravitational equations (2.2) are also satisfied by (3.8) with $\phi(t)$ given by (2.15). It may be shown in the same way as it was done in [1] that, due to the (anti-)self-duality condition, the energy-momentum tensor satisfies

$$T[F, g]_{ij} = 0 \quad \Rightarrow \quad T[F, g]_i^j = 0, \quad (3.10)$$

for all $i, j = 1, \dots, n$, i.e., the form field contributes as dust matter.

4. Generalization to a set of extra Ricci-flat spaces

Here we give a further generalization of the solution of the previous section to the case when the manifold (3.1) is replaced by

$$M = (t_-, t_+) \times N \times N_1 \times \dots \times N_k, \quad (4.1)$$

where N_r are Ricci-flat manifolds with the metric h^r of dimension d_r . The solution reads

$$ds^2 = \exp \left[\frac{4mf(t)}{K(2-D)} \right] \left(-e^{2c^0 t + 2\bar{c}^0} dt^2 + e^{f(t)/K} h_{ij}(y) dy^i dy^j + \sum_{r=1}^k e^{2c^r t + 2\bar{c}^r} ds_r^2 \right), \quad (4.2)$$

$$F = e^{2f(t)} dt \wedge Q, \quad (4.3)$$

where $ds_r^2 = h_{m_r n_r}^r(z_r) dz_r^{m_r} dz_r^{n_r}$ is the line element corresponding to the metric h^r , the constants c^0, \bar{c}^0 are given by

$$c^0 = \sum_{r=1}^k d_r c^r, \quad \bar{c}^0 = \sum_{r=1}^k d_r \bar{c}^r \quad (4.4)$$

and the function $f(t)$ is again given by

$$f(t) = -\ln \left[|z(t)| |KQ^2|^{1/2} \right] \quad (4.5)$$

with

$$z(t) = \frac{1}{\sqrt{C}} \sinh \left[(t - t_0) \sqrt{C} \right], \quad K < 0, \quad C > 0; \quad (4.6)$$

$$= \frac{1}{\sqrt{-C}} \sin \left[(t - t_0) \sqrt{-C} \right], \quad K < 0, \quad C < 0; \quad (4.7)$$

$$= t - t_0, \quad A < 0, \quad C = 0; \quad (4.8)$$

$$= \frac{1}{\sqrt{C}} \cosh \left[(t - t_0) \sqrt{C} \right], \quad K > 0, \quad C > 0. \quad (4.9)$$

Here $D = 4m + 1 + \sum_{r=1}^k d_r$ and

$$K = m + \frac{4m^2}{2-D} \neq 0 \quad (4.10)$$

and the integration constants obey the relation

$$CK^{-1} + \sum_{r=1}^k (c^r)^2 d_r - \left(\sum_{r=1}^k c^r d_r \right)^2 = 0. \quad (4.11)$$

This solution may be readily verified by substitution into the field equations.

As a final remark, the metric (4.2) may also be obtained from the exact cosmological solutions for a one-component anisotropic (“perfect”) fluid [23, 24] when the pressure in the subspace N is $p_0 = 0$ (“dust”) and the pressures in the extra subspaces N_i are $p_i = \rho$ (“stiff” matter), where ρ is the energy density. A systematic derivation of this (and more general) solution will be given in a separate publication.

5. Summary and conclusions

We have considered a $D = (n+1)$ -dimensional cosmological model with an antisymmetric $(p+2)$ -form. We have generalized the composite electric S-brane solutions from [2] for $D = 4m + 1 = 5, 9, 13, \dots$ and $p = 2m - 1 = 1, 3, 5, \dots$ to the case when the Q -form of rank $2m$ is defined on an n -dimensional oriented Ricci-flat space N of Euclidean signature. Here the form Q is an arbitrary (anti-)self-dual parallel (i.e., covariantly constant) $2m$ -form on N with $Q^2 > 0$. For the flat case, when $N = \mathbb{R}^{4m}$ [2], the components of this form in canonical coordinates are proportional to charge densities of electric p -branes.

We have also found generalizations of the solutions to the case when a chain of extra Ricci-flat factor-spaces is added.

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